Fun with Primes

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Let $p_0 = 1, p_1 = 2, \lambda_1 = \{p_1\}, \mathit{\Delta}_1 = 1$ and $\mathit{\Delta}_{n+1} = \mathit{\Delta}_n * p_n$ for $n \in \mathbb{N}$:[1]

$\kappa_n = \{x + i\mathit{\Delta}_n : x \in \lambda_n, i \in \{0, \dots, p_n - 1\}\}$,

$\lambda_{n+1} = \kappa_n \setminus \{p_n * \lambda_n^i : i \in \{0, \dots, \mathit{\Pi}_{i=1}^{n-1}(p_i - 1) - 1\}\} =: \kappa_n \setminus elim(\kappa_n)$, and

$p_n = \lambda_n^1$.

Proposition 1. *For all* $n \in \mathbb{N}, x \in \lambda_n : \frac{x}{p_i} \notin \mathbb{N}$ *for all* $i \in \{1, \dots, n-1\}$.

For all $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the set of prime numbers up to the n-th prime.

Proposition 2. $\mathcal{P}_n = \bigcup_{i=1}^n \lambda_i^1$

We can think of this algorithm as an elimination algorithm. Picture λ as a column vector and κ as it's extension, which basically adds a number of equidistant columns next to the initial one. Then, you run the field alongside growing numbers. The first entry of κ always gets eliminated, then it's square, and so on, all the products of the actual p with all but the last entries of λ.

To see that we don't eliminate too many or too few, we compare the cardinality of $elim(\kappa_n)$ with the n-th prime's $\tilde{p}_n$ naturally occurring elimination rate defined as:

Definition 1. *Let* $a_0 = 0$*, then for all* $n \geq 1$*:* $a_n = \frac{\mathit{\Pi}_{i=1}^{n-1}\tilde{p}_i*(1-\sum_{i=1}^{n-1} a_i)}{\mathit{\Pi}_{i=1}^{n}\tilde{p}_i}$

What we need is this:

Proposition 3. $a_n = \frac{\mathit{\Pi}_{i=1}^{n-1}(\tilde{p}_i-1)}{\mathit{\Pi}_{i=1}^{n}\tilde{p}_i}$

Proof. We proof this by induction. The cases $n = 1, 2, \dots$ can easily be verified. We now assume for an arbitrary but fixed n the claim holds. What we need to show is that $\mathit{\Pi}_{i=1}^n(\tilde{p}_i - 1) = \mathit{\Pi}_{i=1}^n\tilde{p}_i * (1 - \sum_{i=1}^n a_i)$:

$\mathit{\Pi}_{i=1}^n(\tilde{p}_i - 1) = \mathit{\Pi}_{i=1}^{n-1}(\tilde{p}_i - 1) * (p_n - 1) =^{IH} \mathit{\Pi}_{i=1}^{n-1}\tilde{p}_i * (1 - \sum_{i=1}^{n-1} a_i) * (p_n - 1)$

Hence it remains to show:

$\mathit{\Pi}_{i=1}^n\tilde{p}_i * \sum_{i=1}^{n-1} a_i + \mathit{\Pi}_{i=1}^{n-1}\tilde{p}_i * (1 - \sum_{i=1}^{n-1} a_i) = \mathit{\Pi}_{i=1}^n\tilde{p}_i * \sum_{i=1}^n a_i$

This holds, since the following holds:

$\mathit{\Pi}_{i=1}^n\tilde{p}_i * \sum_{i=1}^{n-1} a_i + \mathit{\Pi}_{i=1}^{n-1}\tilde{p}_i * (1 - \sum_{i=1}^{n-1} a_i) = \mathit{\Pi}_{i=1}^n\tilde{p}_i * (\sum_{i=1}^{n-1} a_i + \frac{1-\sum_{i=1}^{n-1} a_i}{p_n}) =$

$= \mathit{\Pi}_{i=1}^n\tilde{p}_i * (\sum_{i=1}^{n-1} a_i + a_n)$

[1] $\mathbb{N} = \{1, 2, \dots\}$. For a set $M = \{x_1, \dots, x_m\}$ we set $M^i = x_i, M^0 = 1, M^+ = x_1$ and $M^- = x_m$.

To see more clearly what happens here, we proof the following proposition:

Proposition 4. *For all natural numbers* $n \geq 2$*:* $\kappa^-_{n-1} * p_n = p_n + \Delta_{n+1}$

Proof. $\kappa^-_{n-1} * p_n = (\lambda^-_{n-1} + \Delta_n - \Delta_{n-1}) * p_n$. This means we have to show $\lambda^-_{n-1} - \Delta_{n-1} = 1$. This holds since for all $m \in \mathbb{N}$:

$$\lambda^-_{m+1} - \Delta_{m+1} = \lambda^-_m + \Delta_m * (p_m - 1) - \Delta_{m+1} = \lambda^-_m - \Delta_m = \lambda^-_1 - \Delta_1 = 1.$$

What this means is that for any prime number p_n, there is a set of corresponding initial primes λ_n which generates a corpus κ_n of potential new primes, all of the candidates being not divisible by any prime smaller than p_n. Then, from κ_n we remove all those candidates that are divisible by p_n, and hence end up with a new λ_{n+1} with the next prime p_{n+1} as initial object. Actually, all elements of λ_m with $\lambda^i_m < p^2_m$ are primes. The interesting thing now is the way how we relate the eliminations to the structure of κ.

Principal Argument:

For any fixed $n \in \mathbb{N}$ it holds that for any $k \in \{1, \cdots |\lambda_n|\}$ there is one unique $x \in elim(\kappa_n)$ such that $x = \lambda^k_n + i\Delta_n$ for some $i \in \{0, \ldots, p_n - 1\}$.

Proof. Let n be fixed and assume there are $r_a, r_b \in \{1\} \cup (\lambda_n \setminus \{\lambda^-_n\})$, with $r_a \neq r_b$ and $x = p_n * r_a = \lambda^k_n + i * \Delta_n, y = p_n * r_b = \lambda^k_n + j * \Delta_n$ for some $k \in \{1, \ldots, |\lambda_n|\}$, and $i, j \in \{0, \ldots, p_n - 1\}$ with $i \neq j$. It then holds:[2]

$$p_n * (r_a - r_b) = \Delta_n * (i - j), \text{ which is the same as to say } \frac{p_n * (r_a - r_b)}{\Delta_n} = i - j.$$

For $0 < r_a - r_b < \Delta_n$, although $r_a - r_b$ can be divisible by factors of Δ_n (as for example by $p_1 = 2$), there must remain at least one factor of Δ_n, since $\frac{r_a - r_b}{\Delta_n} < 1$, but of course p_n cannot be divisible by any such.

What this means is, that if we have s_n (candidate) prime twins in some λ_n we will have $s_n * p_n$ prime twin candidates in κ_n while only $2 * s_n$ can, and will, be eliminated. This leads to the equation $s_{n+1} = s_n * (p_n - 2)$. For example in λ_3 there is one prime twin (candidate), namely 5/7, and in λ_4 there are $1 * (5 - 2) = 3$. In each κ_n there are only numbers not divisible by smaller prime numbers than p_n and those divisible by p_n get eliminated before moving on to λ_{n+1}.

[2] Without loss of generality we assume that $x > y$ and hence $i > j$.