

Game Semantical Rules for Vague Proportional Semi-Fuzzy Quantifiers

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Vague quantifier expressions like “about half” or “almost all”, and their semantics, are of great importance to researchers of various fields, including linguistics, analytic philosophy, mathematical logic and computer science. The aim is to develop adequate models, in order to formally reason with natural language utterances like “At most about a third of all sand corns are black”. While linguists usually address that issue by adopting to the classical principle of two valued logic [10] (true and false), there is a fast growing body of work within the field of mathematical fuzzy logic, building on Zadeh [14], dedicated to the same goal, allowing for intermediate truth values within the real unit interval. This has several reasons, one of which is the desired robustness of the evaluation of formulas. This means, a small change of the information used to evaluate a sentence, may only cause a small change of the corresponding truth value. To that end, we need to require continuity of truth functions for all logical connectives, and the only t-norm based fuzzy logic that can show this behaviour is Lukasiewicz logic.

For this particular logic there is a characterization in terms of a two player zero sum game of perfect information, called Giles’s game (\mathcal{G} -game) [6], where players are called **P** (proponent) and **O** (opponent). In contrast to the more classic Hintikka game [8], in Giles’s game it is possible for both players to have asserted multisets of formulas at each state of the game. This feature results from the following implication rule:

Game Rule 1 (R_{\rightarrow}) *If **P** asserts $F \rightarrow G$ then **O** may attack by asserting F , obliging **P** to assert G .*

In this way, any game state of the form $[F_1, \dots, F_n \mid G_1, \dots, G_m]$, where the F ’s are **O**’s and the G ’s are **P**’s asserted formulas, gets decomposed into a state of the form $[A_1, \dots, A_{n'} \mid B_1, \dots, B_{m'}]$, where the A ’s are atoms which **O** eventually has to take responsibility for and the B ’s are those for which **O** has to account for, while players are supposed to play rationally. Taking up responsibility for an assertion of an atom means to accept having to pay 1€ to the opponent player in case the atom is evaluated to false with respect to a given interpretation I (over a finite domain U with cardinality n) and risk value assignment $\langle \rangle_I$ [1]. Hence, the final risk, from **P**’s perspective, of a game is computed as:

$$\langle A_1, \dots, A_{n'} \mid B_1, \dots, B_{m'} \rangle = \sum_{1 \leq i \leq m'} \langle B_i \rangle_I - \sum_{1 \leq j \leq n'} \langle A_j \rangle_I \quad (1)$$

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The negation of some formula F , defined as $(F \rightarrow \perp)$, introduces role switch of the players, and the following rule for strong conjunction the principle of limited liability:

Game Rule 2 ($R_{\&}$) *If \mathbf{P} asserts $F\&G$ then, if \mathbf{O} attacks, \mathbf{P} has to either assert F as well as G , or else \perp .*

As well as \mathbf{O} need not attack every formula asserted by \mathbf{P} , \mathbf{P} can also hedge his/her loss with regard to asserting more than one formula at once. Here, with the strong conjunction rule, it is stated explicitly that \mathbf{P} can assert \perp instead of both F and G , in case they are both wrong. It be understood that this so called principle of limited liability always remains in place throughout this abstract, although sometimes only implicitly [1]. We can give a characterization of strong Lukasiewicz logic via \mathcal{G} -games as follows:

Theorem 1. [4] *For every atomic formula A let $\langle A \rangle$ be its risk and let I be the L -interpretation given by $v_I(A) = 1 - \langle A \rangle$. Then, if both, \mathbf{P} and \mathbf{O} , play rationally, any game starting in state $\llbracket F \rrbracket$ will end in a state where \mathbf{P} 's final risk is $1 - v_I(F)$.*

In [5], the authors recently proposed a randomized version of this game through introducing a new game rule for a unary quantifier, called Π :

Game Rule 3 (R_{Π}) *If \mathbf{P} asserts $\Pi x F(x)$ then \mathbf{P} has to assert $F(c)$ for a randomly picked c .*

The truth function can be specified as¹:

$$v_I(\Pi x F(x)) = \frac{\sum_{c \in U} v_I(F(c))}{|U|} = Prop_x F(x) \quad (2)$$

Following Liu and Kerre [11], we can distinguish two different sources of vagueness. The one that comes from the attributes, and the one that comes from the semantics of quantifiers. Here, we are going to focus on the latter one, the semantics of quantifiers, and furthermore require their arguments to be classical, a practice supported by Glöckner [7] and others (e.g. Diaz-Hermida, et al. [3]), in order to avoid unclarity regarding the interpretation of statements. Such quantifiers are called semi-fuzzy, and we denote classical formulas with \hat{F} .

One class of semi-fuzzy quantifiers is blind choice quantifiers (see [5]) for which we can derive the following representation:

Theorem 2. $G_m^k = [\Pi]_{\oplus}^{m+1} \& \Pi^{k-1}$ and $L_m^k = G_k^m \neg$

Where $[F]_{\oplus}^r = F \oplus \dots \oplus F$ (strong disjunction), and $F^r = F \& \dots \& F$, r times. For another class of semi-fuzzy quantifiers proposed in [5], so called deliberate choice quantifiers, a similar representation theorem would require a product connective that is originally missing in Lukasiewicz logic, but can be simulated by means of the following game rule²:

¹ We identify constants and domain elements for simplicity.

² \bar{x}, \bar{y} being the truth constants for $x, y \in (0, 1)$.

Game Rule 4 If \mathbf{P} asserts $F \cdot G$ then, if \mathbf{O} attacks, two real numbers $x, y \in (0, 1)$ get chosen randomly, and then \mathbf{P} has to choose $j \in \mathbb{N}$ and assert $[\neg(F \rightarrow \bar{x}) \& \neg(G \rightarrow \bar{y})]_{\oplus}^j$

By using this rule, we can prove the following representation theorem for deliberate choice quantifiers:

Theorem 3. $\Pi_m^k = [\Pi^k \cdot (\neg\Pi)^m]_{\oplus}^{\binom{k+m}{m}}$

Since with the Π quantifier sampling of constants (domain elements) is done with replacement, we cannot reduce the following unary quantifier rule to the Π rule³:

Game Rule 5 ($R_{\Pi^{j,k}}$) If \mathbf{P} asserts $\Pi^{j,k} x \hat{F}(x)$ then, if \mathbf{O} attacks, \mathbf{P} has to assert $\hat{F}(c)$ for j of k different and uniformly chosen c 's, or else \perp .

We claim this rule matches the following truth function ($p = Prop_x \hat{F}(x)$):

$$v_I(\Pi^{j,k} x \hat{F}(x)) = \frac{\sum_{i=j}^k \binom{np}{i} \binom{n-np}{k-i}}{\binom{n}{k}} \quad (3)$$

Theorem 4. A $L(R_{\Pi^{j,k}})$ -sentence $\Pi^{j,k} x \hat{F}(x)$, for a classical formula \hat{F} is evaluated to $v_M(\Pi^{j,k} x \hat{F}(x)) = x$ in an interpretation I iff every \mathcal{G} -game for $\Pi^{j,k} x \hat{F}(x)$ augmented by rule ($R_{\Pi^{j,k}}$) is $(1-x)$ -valued for \mathbf{P} under risk value assignment $\langle \cdot \rangle_I$.

We can use this analytic rule to model vague proportional quantifiers:

Semi-Fuzzy Quantifier	Intended Meaning
$\Pi^{j,j+1}$	almost all
$\Pi^{j,j+1} \neg$	nearly none
$\Pi^j, \lfloor n(1 - \frac{x}{100}) \rfloor + j$	at least about $x\%$
$\Pi^j, \lfloor \frac{nx}{100} \rfloor + j \neg$	at most about $x\%$
$(\Pi^j, \lfloor n(1 - \frac{x}{100}) \rfloor + j) \& (\Pi^j, \lfloor \frac{nx}{100} \rfloor + j \neg)$	approximately $x\%$

Another way of conceiving the semantics of the quantifier “approximately $x\%$ ”, then called “about $x\%$ ”, is by means of a strong disjunction, rather than a strong conjunction, as we might accept the meaning of, say, “about half” to be “a bit more than half, or a bit less than half”. To this end we define the following quantifier:

Definition 1. For $s \leq n$ and $A, B \in \mathcal{P}(\{1, \dots, s\}) \setminus \emptyset$:

$$Q^{s,A,B} = (\bigoplus_{i \in A} (\neg(\Pi^{i,s} \rightarrow \Pi^{i+1,s}))) \oplus (\bigoplus_{i \in B} (\neg(\Pi^{i,s} \rightarrow \Pi^{i+1,s})))$$

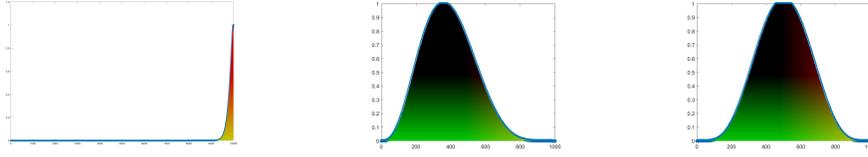


Figure 1. left: truth function of “almost all” modeled by $\Pi^{100,10^1}$, $n = 10000$; middle: truth function of “about a third” modeled by $Q^{12,\{3,4\},\{4,5,6\}}$, $n = 1000$; right: truth function of “about half” modeled by $Q^{14,\{5,6,7\},\{7,8,9\}}$, $n = 1000$.

A non-trivial analysis of the sets A and B yields a parametrization for intuitive models of the quantifiers “about $x\%$ ”, which are then only dependent on s (and x of course). Figure 1 shows truth functions of some particular quantifiers based on the $\Pi^{j,k}$ rule.

Future work comprises the handling of the fully-fuzzy scenario, where the arguments of quantifiers may be fuzzy (multi valued) formulas. To achieve this we employ a general lifting mechanism, fulfilling a certain list of axioms resulting from first principles of reasoning, as proposed by Glöckner [7]. Also, we will extend the described setting by connecting it to independence friendly logic, which fits games of imperfect information [12].

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³ Note that $\Pi^{1,1} = \Pi$.

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