Fun with Primes

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Let $p_0 = 1, p_1 = 2, \lambda_1 = \{p_1\}, \Delta_1 = 1$ and $\Delta_{n+1} = \Delta_n \ast p_n$ for $n \in \mathbb{N}$.

$k_n = \{x + i\Delta_n : x \in \lambda_n, i \in \{0, \ldots, p_n - 1\}\}$,

$\lambda_{n+1} = k_n \setminus \{p_n \ast \lambda_n^i : i \in \{0, \ldots, \Pi_{i=1}^{n-1}(p_i - 1)\}\}$ =: $\kappa_n \setminus \text{elim}(\kappa_n)$, and $p_n = \lambda_n^1$.

**Proposition 1.** For all $n \in \mathbb{N}, x \in \lambda_n : \frac{x}{p_i} \not\in \mathbb{N}$ for all $i \in \{1, \ldots, n - 1\}$.

For all $n \in \mathbb{N}$, let $P_n$ be the set of prime numbers up to the $n$-th prime.

**Proposition 2.** $P_n = \bigcup_{i=1}^{n} \gamma_i^1$

We can think of this algorithm as an elimination algorithm. Picture $\lambda$ as a column vector and $\kappa$ as it’s extension, which basically adds a number of equidistant columns next to the initial one. Then, you run the field alongside growing numbers. The first entry of $\kappa$ always gets eliminated, then it’s square, and so on, all the products of the actual $p$ with all but the last entries of $\lambda$.

To see that we don’t eliminate too many or too few, we compare the cardinality of $\text{elim}(\kappa_n)$ with the $n$-th prime’s $\tilde{p}_n$ naturally occurring elimination rate defined as:

**Definition 1.** Let $a_0 = 0$, then for all $n \geq 1$: $a_n = \frac{\Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} a_i)}{\Pi_{i=1}^{n-1} \tilde{p}_i}$

What we need is this:

**Proposition 3.** $a_n = \frac{\Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} a_i)}{\Pi_{i=1}^{n-1} \tilde{p}_i}$

**Proof.** We proof this by induction. The cases $n = 1, 2, \ldots$ can easily be verified. We now assume for an arbitrary but fixed $n$ the claim holds. What we need to show is that $\Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} a_i)$:

$\Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} a_i) = \Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} \frac{a_i}{p_n})$

This holds, since the following holds:

\[
\Pi_{i=1}^{n-1} \tilde{p}_i \ast (1 - \sum_{i=1}^{n-1} \frac{a_i}{p_n}) = \Pi_{i=1}^{n-1} \tilde{p}_i \ast (\sum_{i=1}^{n-1} a_i + 1 - \sum_{i=1}^{n-1} \frac{a_i}{p_n}) = \]

\[
\Pi_{i=1}^{n} \tilde{p}_i \ast (\sum_{i=1}^{n-1} a_i + a_n)
\]

$^1 \mathbb{N} = \{1, 2, \ldots\}$. For a set $M = \{x_1, \ldots, x_m\}$ we set $M^1 = x_1, M_0 = 1, M^+ = x_1$ and $M^- = x_m$. 
To see more clearly what happens here, we proof the following proposition:

**Proposition 4.** For all natural numbers \( n \geq 2 \): \( \kappa_{n-1}^n * p_n = p_n + \Delta_{n+1} \)

**Proof.** \( \kappa_{n-1}^n * p_n = (\lambda_{n-1}^n + \Delta_n - \Delta_{n-1}) * p_n \). This means we have to show \( \lambda_{n-1}^n - \Delta_{n-1} = 1 \). This holds since for all \( m \in \mathbb{N} \):

\[
\lambda_{m+1}^n - \Delta_{m+1} = \lambda_m^n + \Delta_m^* (p_m - 1) - \Delta_{m+1} = \lambda_m^n - \Delta_m = \lambda_1^* - \Delta_1 = 1.
\]

What this means is that for any prime number \( p_n \), there is a set of corresponding initial primes \( \lambda_n \) which generates a corpus \( \kappa_n \) of potential new primes, all of the candidates being not divisible by any prime smaller than \( p_n \). Then, from \( \kappa_n \) we remove all those candidates that are divisible by \( p_n \), and hence end up with a new \( \lambda_{n+1} \) as initial object. Actually, all elements of \( \lambda_m \) with \( \lambda_m^i < p_m^2 \) are primes. The interesting thing now is the way how we relate the eliminations to the structure of \( \kappa \).

**Principal Argument:**

For any fixed \( n \in \mathbb{N} \) it holds that for any \( k \in \{1, \ldots, |\lambda_n|\} \) there is one unique \( x \in \text{elim}(\kappa_n) \) such that \( x = \lambda_n^k + i \Delta_n \) for some \( i \in \{0, \ldots, p_n - 1\} \).

**Proof.** Let \( n \) be fixed and assume there are \( r_a, r_b \in \{1\} \cup (\lambda_n \setminus \{\lambda_n^k\}) \), with \( r_a \neq r_b \) and \( x = p_n * r_a = \lambda_n^k + i * \Delta_n, y = p_n * r_b = \lambda_n^k + j * \Delta_n \) for some \( k \in \{1, \ldots, |\lambda_n|\} \), and \( i, j \in \{0, \ldots, p_n - 1\} \) with \( i \neq j \). It then holds:

\[
p_n * (r_a - r_b) = \Delta_n * (i - j),
\]

which is the same as to say \( \frac{p_n * (r_a - r_b)}{\Delta_n} = i - j \).

For \( 0 < r_a - r_b < \Delta_n \), although \( r_a - r_b \) can be divisible by factors of \( \Delta_n \) (as for example by \( p_1 = 2 \)), there must remain at least one factor of \( \Delta_n \), since \( \frac{r_a - r_b}{\Delta_n} < 1 \), but of course \( p_n \) cannot be divisible by any such.

What this means is, that if we have \( s_n \) (candidate) prime twins in some \( \lambda_n \), we will have \( s_n * p_n \) prime twin candidates in \( \kappa_n \) while only \( 2 * s_n \) can, and will, be eliminated. This leads to the equation \( s_{n+1} = s_n * (p_n - 2) \). For example in \( \lambda_3 \) there is one prime twin (candidate), namely \( 5/7 \), and in \( \lambda_4 \) there are \( 1 * (5 - 2) = 3 \). In each \( \kappa_n \) there are only numbers not divisible by smaller prime numbers than \( p_n \) and those divisible by \( p_n \) get eliminated before moving on to \( \lambda_{n+1} \).

\[2\] Without loss of generality we assume that \( x > y \) and hence \( i > j \).