section 1, Introduction

The problem we are concerned with deals with independent and identically distributed Rademacher-$(\frac{1}{2})$ random variables $X_i$, $i \in \mathbb{N}$, with values in $\{-1, +1\}$. For fixed $n \in \mathbb{N}$ we can understand a realization of the random vector $\mathbf{X}_n := (X_1, \ldots, X_n)$ as a lattice path on $\mathbb{N}_0 \times \mathbb{Z}$. We are interested in the joint distribution of the following three random variables:

$$S_n := \sum_{i=1}^n X_i$$
$$A_n := \sum_{i=1}^n S_i$$
$$\tau(-1) := \min \{i \in \mathbb{N} : S_i = -1\}$$

$(S_0 = A_0 = 0)$

So we fix a probability space $(\Omega_n, \mathcal{P}(\Omega_n), \mathbb{P})$ as follows:

$\Omega_n := \{\{z_1, \ldots, z_n\} : \forall i \in \{1, \ldots, n\} \text{ it holds } z_i \in \{-1, +1\}\}$, $\mathcal{P}(\Omega_n)$ is the power-set of $\Omega_n$, and $\mathbb{P}$ the uniform probability measure. The term in question now is:

$$\mathbb{P}(A_n \geq x, \tau(-1) > n \mid S_n = y) \quad \text{for } n \in \mathbb{N}, \ y \in \mathbb{R}_0^+ \text{ and } x \in \mathbb{R}^+.$$ 

Graphicly speaking, we are interested in how many paths, with fixed start- and endpoint, and the further condition to stay non-negative always, lead to the same area locked up between the path and the $x -$ axis. We use the following decomposition of our term in question:

$$\mathbb{P}(A_n = t, \tau(-1) > n \mid S_n = y) = \mathbb{P}(A_n = t \mid S_n = y) - \mathbb{P}(A_n = t, \tau(-1) \leq n \mid S_n = y) =$$

$$= \mathbb{P}(A_n = t \mid S_n = y) - \sum_{i=1}^n \mathbb{P}(A_n = t, \tau(-1) = i \mid S_n = y) =$$

$$= \mathbb{P}(A_n = t \mid S_n = y) - \frac{1}{\mathbb{P}(S_n = y)} \sum_{i=1}^n \sum_{\varphi \in \Phi(i)} \mathbb{P}(A_i = \varphi, \tau(-1) = i) \mathbb{P}(\tilde{A}_{n-i} = t - \varphi, \tilde{S}_{n-i} = y),$$

(8)

with $t \in \Psi(n)$ and $\tilde{S}_0 = -1$. While we use this notation:

$$x_n^+ := \max \{\zeta \in \mathbb{N} : \exists \omega \in \Omega_n \text{ mit } A_n(\omega) = \zeta, S_n(\omega) = y\} = \frac{1}{\tau}(n^2 - y^2 + 2y + 2ny)$$
$$x_n^- := \min \{\zeta \in \mathbb{N} : \exists \omega \in \Omega_n \text{ mit } A_n(\omega) = \zeta, S_n(\omega) = y\} = \frac{1}{\tau}(y^2 - n^2 + 2y + 2ny),$$

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\[ \theta_i := \frac{i-1}{2}, \] where the interesting values of \( i \) are only the uneven numbers smaller than \( n \).

\[ z_i^+ := \max \{ A_i : \tau(-1) = i \} = \theta_i^2 - 1, \] and \( z_i^- := \min \{ A_i : \tau(-1) = i \} = \theta_i - 1, \)

\[ \Phi(i) := \left\{ z_i^+ - 2\vartheta : \vartheta = 0, 1, 2, \ldots, \left(\frac{\theta_i}{2}\right) \right\}, \] and \( \Psi(n) := \{ x_n^+ - 2\vartheta : \vartheta = 0, 1, 2, \ldots, x_n^+ \}. \)

The following picture visualizes the situation.

In the terms of (\( \mathcal{K} \)), without the random variable \( \tau(-1) \), we find as common ground that the possible paths have the same start- and endpoint without further restrictions, and thus their distribution only depends on the pathlength and the ups and downs, meaning \( \text{ups} = u := \frac{n+y}{2} \) and \( \text{downs} = d := \frac{n-y}{2} \). We will investigate this distribution generally in the following section, depending on the values \( n \) and \( y \). The remaining terms get treated separately in section 3.

**section 2, Integer Partitions**

For all \( n \in \mathbb{N} \) we have an unique \( \omega \in \Omega_n \) with \( A_n(\omega) = x_n^+ \). Since we have for all \( \omega \in \Omega_n \), that \( A_n(\omega) \in \Psi(n) \), we need to understand how the deviation of \( A_n(\omega) \) from \( x_n^+ \) can be used to determining the probability \( P(A_n = x_n^+ - 2\vartheta|S_n = y) \). Due to brevity and clarity we will focus on the special case, where \( y = 0 \), and try to convince the reader that the general case is also covered by the presented strategy, which gets illustrated by the following picture.
\( k = 10 \)
\( n = 2 \)
\( x_n + = k^2 \)
\( u = d = k \)
\( x_n^\prime = 2r \) with \( r = \sum_{i=0}^{u-1} (u-i) a_i \)

where \( a_i \) corresponds to the number of consecutive downs after a total of \( i \) ups.

here: \( a_5 = 2, a_7 = 1, a_8 = 2, \) and \( a_9 = 3 \)

By renaming \((\#\equiv a_{u-j})\) we arrive at the following representation of \( r \):

\[
r = \sum_{i=0}^{u-1} (k - i) a_i = \sum_{j=1}^{u} j \#^j = \#^1 + 2 \#^2 + \ldots + u \#^u
\]

Since each value of \( r \) corresponds to a number of paths \( \omega \in \Omega_n \) with \( S_n(\omega) = y \), we have generally to fulfill the extra condition \( \sum_{j=1}^{u} \#^j \leq d \).

\[
\text{IP}(r) := \{ (\#^1, \ldots, \#^r) \in \{0, \ldots, r\}^r : r = \sum_{j=1}^{r} j \#^j \} \text{ is the set of integer partitions of an integer } r.
\]

Generally we are interested in the following function: (for the special case \( y = 0 \), we have \( u = d = k \))

\[
\xi: \mathbb{N}^3 \to \mathbb{N} \text{ with } \xi(r, u, d) = \left| \{ (\#^1, \ldots, \#^u) \in \{0, \ldots, d\}^u : r = \sum_{j=1}^{u} j \#^j, \sum_{j=1}^{u} \#^j \leq d \} \right|
\]

Also, \(|\text{IP}(r)| = \xi(r, r, r)\), and asymptotically we have \(|\text{IP}(r)| \sim \frac{e^{\sqrt{\pi r}}}{4r^{3/2}} \). \[1\]

An exact formula for \(|\text{IP}(r)|\) is rather involved and was presented by Hans Rademacher in 1937. \[1\]

We arrive at \( \xi(r, u, d) \) through splitting the cases:

\textit{subsection 2.1, } \( \xi(r, u, r) \)

We consider the generating function \( R \) for integer partitions which is defined for \( |x| < 1 \). Furthermore we use this notation:

\[
a_n(r) := \xi(r, u, r) \text{ and } q(j, i) := 1_{\{i=0\} \cup \{j \in \mathbb{N}\}}.
\]

Of course we have for \( p, q \in \mathbb{N} \) with \( p \leq q \), that \( \xi(p, q, p) = \xi(p, p, p) \). Furthermore we have \( \forall m \in \mathbb{N} a_m(0) = 1 \), but \( \forall l \in \mathbb{N} \) it is \( a_0(l) = 0 \), and follow the convention \( a_0(0) = 1 \).
\[ R(u) := \prod_{m=1}^{u} \frac{1}{(1-x^m)} = \prod_{m=1}^{u} \sum_{j=0}^{\infty} q(j, m) x^j = \sum_{j=0}^{\infty} a_u(j) x^j \]

So, for \( t \in \mathbb{N} \)
\[ \sum_{j=0}^{\infty} q(j, t) x^j = \sum_{j=0}^{\infty} a_1(j) x^j, \quad \text{and} \]
\[ \sum_{j=0}^{\infty} a(t) x^j = \sum_{j=0}^{\infty} a_{t-1}(j) x^j \sum_{j=0}^{\infty} q(j, t) x^j, \quad \text{and therefore} \]
\[ a_t(i) = \sum_{l=0}^{i} q(l, t) a_{t-1}(i-l) \]

Hence we can conclude, that for all \( t \in \mathbb{N} \) and \( i \in \mathbb{N}_0 \) there are numbers \( a_t^1, a_t^i \in \mathbb{Q}_0^+ \), such that
\[ a_t(i) := a_t^{i-1}i^{-1} + a_t^{i-2}t^{-2} + \ldots + a_t^0 - a_t^i, \quad \text{e.g.} \]
\[ a_t(1) = a_t^0 \]
\[ a_t^2(i) = \frac{i}{2} + 1 - \sigma_{t, 2} a_t^0 + a_t^2 - a_t^i \]
\[ a_t^3(i) = \frac{i^2}{12} + \frac{7i}{12} + 1 - \left( \sigma_{t, 3}/4 + (3\sigma_{t, 2}^2)/4 + \sum_{l=0}^{i/3} \sigma_{t-3l, 2} \right) = a_t^3 i^2 + a_t^3 i + a_t^0 - a_t^i, \quad \text{where we use the} \]
notation \( \sigma_{t, 1, t} := \left[ \frac{i}{t} \right] (i \text{ mod } t) \), for which holds \( \left\lfloor \frac{i}{t} \right\rfloor = \frac{i}{t} - \sigma_{t, 1, t} \), with \( \lfloor x \rfloor \) being the \( \lfloor x \rfloor \) \text{ integers floor function}. \text{One can derive, that} \( a_t^{i-1} = \frac{1}{v(t-1)!} \) and \( a_t^{i-2} = \frac{(1+t)}{4(t-1!)(t-2)!} \).

\textbf{Theorem 1:}

For all \( t, r \in \mathbb{N} \) with \( t < r \) it holds:
1. \( a_r(r) = \sum_{j=1}^{r} a_j(r-j) \)
2. \( a_r(r) = a_r(r) - \sum_{m=0}^{r-1} a_r-1-m(m) \).

\textbf{Corollary:}

1. For \( t < r \leq 2t+1 \) we have:
\[ a_r(r) = a_r(r) - \sum_{m=0}^{r-1} a_r(m) \]
2. For \( 2t+2 \leq r \leq 3t+2 \) we have:
\[ a_r(r) = a_r(r) - \sum_{m=0}^{r-1} a_r(m) - \sum_{m=[r/2]+1}^{r-1} a_r(m) - \sum_{j=0}^{2m-r} a_r(j) \).

\textbf{subsection 2.2, \( \xi(r, u, d) \)}

\[ \xi(r, u, d) = \xi(r, u, r) - \sum_{(\#^1, \ldots, \#^u) \in \{0, \ldots, r\}^u} r = \sum_{j=1}^{u} j \#^j, \sum_{j=1}^{u} \#^j > d \] := \( \xi(r, u, r) - g(r, u, d) \)

\textbf{Theorem 2:}

a. Generally for \( r \geq 2t+1 \):
\[ \xi(r, t, l) = \xi(r, t, r) - g(r, t, l) = \xi(r, t, r) - \sum_{l=0}^{l-1} a_t(l) - \sum_{l=k}^{r-1} \xi(l, t-1, r-l) \]
b. In particular:
1. For \( r \in \{ t+1, \ldots, 2t \} \) we have:
\[ \xi(r, t, l) = \xi(r, t, r) - g(r, t, l) = a_r(r) - \sum_{m=0}^{r-1} a_m(m) \]
2, for \( r \in \{2t+1, \ldots, 3t+1\} \) we have:
\[
\xi(r, t, l) = \bar{a}(r) - \sum_{l=0}^{r-1} a(l) - \sum_{l=k}^{r-1} (a_{l-1}(l) - \sum_{m=0}^{2l-r-1} a_{l-2}(m)).
\]

The most general setting with \( r > d \) and \( u > d \) can be treated with the same technique used in the proof of Theorem 2, but the case distinction then becomes a bit longish as do the formulas.

### section 3, non-negative paths

We consider the Index \( i \) in \((\mathfrak X)\) fixed as well as the value of \( j \in \{0, 1, 2, \ldots, \binom{d}{2}\}\), and work with \( \theta \equiv \theta_i \).

The number of possible paths here is \( C_\theta := \binom{2\theta}{\theta_\theta + 1}, \) while these numbers are known as Catalan numbers, for \( \theta \in \mathbb{N} \). The ones leading to \( A_i = z_1^i - 2j \) is captured by the term \( F_\theta(\theta + 2j) \), for which we introduce its generating function: \( \varphi_\theta(z) = \sum_{j=0}^{\binom{d}{2}} F_\theta(\theta + 2j) z^j \).

**Theorem 3:** [2]

It holds, that \( \varphi_m(z) = \sum_{i=1}^{m} \varphi_{m-i}(z) \varphi_{i-1}(z) z^{i-1} \) für \( m = 1, 2, \ldots \) und \( \varphi_0(z) = 1 \).

### section 4, consolidation

**Theorem 4:**

\[
\mathbb{P}(A_n = x_n^+ - 2r, \tau(-1) > n | S_n = y) = \frac{\xi(r, u, d)}{\binom{d}{2}} - \sum_{i=1}^{n-1} \mathbb{P}(\mathbb{N}_{\mathbb{N}}) \sum_{i \in \mathbb{N}} \left( \frac{F_\theta(\xi(i, r, u, d)) \mathbb{P}(\tau(\xi(i, r, u, d) \in \mathbb{N}))}{\binom{d}{2}} \right)
\]

using this notation:

1. \( n - i = \bar{u} + d \) and \( y + 1 = \bar{u} - d \), hence \( \bar{u} = \frac{n - i + y + 1}{2} \) and \( d = \frac{n - i - y - 1}{2} \)
2. \( x_n^+ = \bar{u} (\bar{u} - 2) + \frac{1}{2} (y - y^2) + 1 \) and \( x_n^- = d (d - 2) + \frac{1}{2} (y + y^2) \)
3. \( G_n := \{0, 1, \ldots, \frac{x_n^+ - x_n^-}{2}\} \), and 4. \( r(\varphi) = \frac{x_n^+ - x_n^- + 2r + \varphi}{2} \).

When we have \( y = 0 \), we can assume without loss of generality, that \( r \in \{0, 1, \ldots, \frac{k^2 - k}{2}\} \), since otherwise \((\mathfrak X)\) is a priori 0.

**Theorem 5:**

For \( r = \mu k^2 \) and \( \mu < \left( \frac{\ln (4)}{\pi \sqrt{2}} \right)^2 \approx 0.292 \) we have:

\[
\mathbb{P}(A_n \geq x_n^+ - 2r, \tau(-1) > n | S_n = 0) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

References: